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Foundations of Phase-Space Quantum Mechanics[1]

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In the present paper a general concept of a phase-space representation of the ordinary Hilbert-space quantum theory is formulated, and then, by using some elementary facts of functional analysis, several equivalent forms of that concept are analyzed. Several important physical examples are presented in Section 3 of the paper.

1. INTRODUCTION

The problem of reformulating the Hilbert-space quantum mechanics in the classical phase space, that is, reexpressing the quantum-mechanical mean values as classical averages over phase-space distribution functions, has received a considerable amount of attention from both physicists and mathematicians, and has a long history (see, e.g., Wigner, 1932, 1971; Husimi, 1940; Groenewold, 1946; Moyal, 1949; Weyl, 1950; Bopp, 1956; Margenau and Hill, 1961; Segal, 1961; Mehta, 1964, 1965; Pool, 1966; Cohen, 1966a, b; Misra and Shankara, 1968; Agarwal and Wolf, 1970; Cushen and Hudson, 1971; Hudson, 1974; Srinivas and Wolf, 1975; O'Connell and Wigner, 1981a, b). For a review of early attempts of the phase-space representation of quantum mechanics, see Sudarshan (1962).

It is known, however, from a result due to Wigner (1932, 1971) that these distribution functions cannot in general be nonnegative, so they are

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not true probability distributions. In this connection, see also the papers by Cohen (1966b) and by Margenan and Cohen (1968), where it has been shown that quantum mechanics cannot be formulated as a classical statistical theory in phase space. Nevertheless, the phase-space representation of quantum mechanics, giving us the possibility of expressing the quantum-mechanical expectations as phase-space integrals, has proven to be very useful in the investigation of many physical problems, particularly in statistical mechanics (see Mori et al., 1962) and in the study of the coherent properties of light (see, e.g., Mandel and Wolf, 1965).

It should be noted that nonnegative distribution functions have also been considered in the literature (see, for example, Husimi, 1940; Bopp, 1956; Kano, 1965; Mehta and Sudarshan, 1965); however, they too cannot be considered as true joint probability distributions for position and momentum, which is clear from the above-mentioned result due to Wigner.

An important step in resolving this "positive-definiteness dilemma" has recently been taken within the so-called "stochastic phase space" scheme developed by Ali and Prugovečki (1977a, b, c), Prugovečki (1978a,b,c,1979,1981a), and Ali, Gagnon, and Prugovečki (1981). This new approach is based on the observation that the concept of the ordinary phase space, as consisting of "sharp" points, is meaningless from the experimental point of view, since it is experimentally impossible to measure phase-space points to more accuracy than the smallest possible marker of a point, an elementary (massive) particle. So, the unavoidable conclusion of the "stochastic phase space" approach is that the concept of the orthodox phase space consisting of sharp points has to be replaced by the more realistic one, whose "point" is identified with the mean position and momentum of the elementary particle indexing it.

The stochastic phase space approach has proven to be very useful in solving some old problems of the relativistic quantum theory. It leads, for instance, to well-defined relativistic quantum currents (Prugovečki, 1978d; Ali et al., 1981), and removes many divergencies which are characteristic for the ordinary quantum mechanics (Prugovečki, 1981a, b). Moreover, there are interesting connections between the stochastic phase space approach and the Weyl correspondence (Schroeck, 1981, 1982). Thus, it would be desirable to have a mathematically precise formulation of the stochastic phase-space representation of quantum mechanics or, even more generally, of the concept of a general phase-space representation of quantum mechanics.

The main aim of the present paper is to give such a rigorous formulation of the notion of the phase-space representability of quantum mechanics by utilizing the concepts of functional analysis, and then to show several equivalent forms of that formulation. Some physical examples illustrating the general theory are presented in Section 3 of the paper.

2. THE GENERAL CONCEPT OF A PHASE-SPACE REPRESENTATION OF QUANTUM MECHANICS. BASIC THEOREMS

By a phase-space representation of quantum mechanics we usually mean any, presumably one-one, affine (i.e., preserving convex combinations) map

$$F: w \mapsto F_w \tag{1}$$

of the convex set of quantum states (consisting of the density operators on the Hilbert space H corresponding to a given physical system) into the classical probability densities on the phase space Γ, together with the linear map

$$A: f \mapsto A(f) \tag{2}$$

from the classical to quantum bounded observables, which preserves the mean values of observables in the sense that

$$\langle A(f); w\rangle_q = \langle f; F_w\rangle_c \tag{3}$$

where $\langle\,;\,\rangle_q$ and $\langle\,;\,\rangle_c$ denote the quantum and the classical mean value, respectively, that is

$$\langle A(f); w\rangle_q = \mathrm{Tr}(A(f)w) \tag{4}$$

$$\langle f; F_w\rangle_c = \int_\Gamma f(q,p)F_w(q,p)\,dq\,dp \tag{5}$$

where Γ denotes the phase space $\mathbb{R}^{2n}$ (corresponding to a physical system with n degrees of freedom), and $dq\,dp$ stands for the Lebesgue measure on Γ.

It can be shown, however, that *the existence of the linear mapping A need not be postulated separately, since it follows as a consequence of* (1).

To prove this, we shall first note that the affine map F can be immediately extended to an affine map $\tilde{F}$ of the positive cone $B^1(H)_+$ of $B^1(H)$, the Banach space of the trace-class operators on H, into the corresponding positive cone $L^1(\Gamma)_+$ of $L^1(\Gamma)$, the latter denoting the Banach space of Lebesgue-integrable complex-valued functions on Γ, by setting for an arbitrary $u \in B^1(H)_+$

$$\tilde{F}(u) = \begin{cases} \|u\|_1 F_{u/\|u\|_1} & \text{if } u \neq 0 \\ 0 & \text{if } u = 0 \end{cases} \tag{6}$$

where $\|\cdot\|_1$ denotes the trace-norm in $B^1(H)$ ($\|u\|_1 = \mathrm{Tr}(u^*u)^{1/2}$), and then $\tilde{F}$ can be uniquely extended to a linear operator acting on the entire space $B^1(H)$ by letting for an arbitrary $v = u_1 - u_2 + i(u_3 - u_4) \in B^1(H)$, where $u_i \in B^1(H)_+$, $i = 1, 2, 3, 4$,

$$\tilde{F}(v) = \tilde{F}(u_1) - \tilde{F}(u_2) + i\tilde{F}(u_3) - i\tilde{F}(u_4) \tag{7}$$

It can be easily seen that the above definition does not depend on any particular choice of the positive elements u_i in the decomposition of v. Moreover, since $\tilde{F}$, when restricted to the positive cone $B^1(H)_+ \subseteq B^1(H)$, is norm preserving:

$$\|\tilde{F}(u)\|_{L^1(\Gamma)} = \|u\|_1, \qquad u \in B^1(H)_+ \tag{8}$$

one easily concludes that $\tilde{F}$ has to be a norm-contracting map of $B^1(H)$ into $L^1(\Gamma)$:

$$\|\tilde{F}(v)\|_{L^1(\Gamma)} \leqslant \|v\|_1 \tag{9}$$

for all v in $B^1(H)$. This means that $\|\tilde{F}\| \leqslant 1$, where $\|\cdot\|$ stands for the usual operator norm of

$$\tilde{F}\left[\|\tilde{F}\| = \sup\left\{\|\tilde{F}(v)\|_{L^1(\Gamma)} : v \in B^1(H), \|v\|_1 = 1\right\}\right]$$

Remark. It is not difficult to check that if F is one-to-one, then its (unique) linear extension $\tilde{F}$ is one-to-one too, and furthermore

$$\|\tilde{F}(v)\|_{L^1(\Gamma)} = \|v\|_1$$

for all $v \in B^1(H)$, so that $\tilde{F}$ is then an isometry.

Now, keeping in mind the facts that the Banach duals of $L^1(\Gamma)$ and $B^1(H)$ are isometrically isomorphic to $L^\infty(\Gamma)$ and $B(H)$, respectively (see, for example, Dunford and Schwartz, 1958; Schatten, 1960), where $L^\infty(\Gamma)$ denotes, as usual, the vector space of essentially bounded complex-valued Lebesgue-measurable functions on Γ, and $B(H)$ denotes the algebra of bounded linear operators on Γ, we may consider the Banach dual $\tilde{F}^*$: $L^1(\Gamma)^* \to B^1(H)^*$ of the linear map $\tilde{F}$ as acting from $L^\infty(\Gamma)$ into $B(H)$. It is, clearly, a positive norm-contracting map too. Moreover, since the isometric isomorphisms

$$B(H) \cong B^1(H)^*$$

$$L^\infty(\Gamma) \cong L^1(\Gamma)^*$$

have been established by using the dualities

$$\langle B, v \rangle = \mathrm{Tr}(Bv), \qquad B \in B(H), \qquad v \in B^1(H)$$

$$\langle f, g \rangle = \int_\Gamma f(q,p) g(q,p)\, dq\, dp, \qquad f \in L^\infty(\Gamma), \qquad g \in L^1(\Gamma)$$

we see that $A(f)$ can be identified with $\tilde{F}^*(f)$, so that

$$A = \tilde{F}^* \tag{10}$$

as claimed.

We are now prepared for introducing a precise notion of a phase-space representation of quantum mechanics. We replace, for generality, the correspondence (1) by a more general one

$$\mu\colon w \mapsto \mu_w \tag{11}$$

where μ_w is a probability measure on Γ, not necessarily absolutely continuous with respect to the Lebesgue measure.

By a *phase-space representation of quantum mechanics* we shall mean any (not necessarily one-to-one) affine map μ: $w \mapsto \mu_w$ of the convex set of the density operators acting on a given Hilbert space H into the convex set of the probability measures on the phase space Γ.

It is clear that applying essentially the same extension procedure as before, we can extend the map μ to a unique positive norm-contracting linear transformation

$$\tilde{\mu}\colon B^1(H) \to M(\Gamma)$$

where $M(\Gamma)$ denotes the space of all bounded complex measures on Γ, which is a Banach space under the norm it inherits as the dual of $C_0(\Gamma)$, the space of all bounded complex-valued functions on Γ which vanish at infinity (for definitions see, e.g., Dunford and Schwartz, 1958). Moreover, if μ is one-to-one, then $\tilde{\mu}$ is obviously one-to-one too, so that $\tilde{\mu}$ is then easily seen to be an isometry.

Note that for a fixed Borel subset $E \subseteq \Gamma$, the map

$$v \mapsto [\tilde{\mu}(v)](E) \tag{12}$$

is clearly a positive linear functional on $B^1(H)$, and therefore continuous, so that there exists a bounded positive linear operator $x(E)$ on H such that

(see, e.g., Schatten, 1960)

$$\mathrm{Tr}(x(E)v) = [\tilde{\mu}(v)](E) \tag{13}$$

for every $v \in B^1(H)$. It is an easy matter to check that x is a POV measure (positive operator-valued measure) on $B(\Gamma)$, the σ-algebra of Borel subsets of Γ.

Conversely, if x is a POV-measure on $B(\Gamma)$, then one can define the phase-space representation mapping by

$$w \mapsto \mathrm{Tr}(x(\cdot)w), \qquad w \in S \tag{14}$$

where S stands for the set of all density operators on H.

We thus see that a *phase-space representation of quantum mechanics may alternatively be described by choosing a particular POV-measure on* $B(\Gamma)$.

Before we go further, we need some definitions. We shall say that the phase-space representation μ: $w \mapsto \mu_w$ is *absolutely continuous* if for any quantum-mechanical state $w \in S$ its associated measure μ_w is absolutely continuous with respect to the Lebesgue measure on Γ.

We call μ *nondegenerate* if it is one-to-one. Otherwise, μ is said to be *degenerate*.

Similarly, a POV-measure x: $B(\Gamma) \to B(H)_+$ is said to be absolutely continuous (respectively, nondegenerate or degenerate) if so is its corresponding phase-space representation map (14).

In the sequel, we shall restrict ourselves to the case of an absolutely continuous phase-space representation μ: $w \mapsto \mu_w$, and denote by F_w the Radon–Nikodým derivative of μ_w with respect to the Lebesgue measure on Γ.

Since μ_w is a nonnegative measure, F_w is a.e. nonnegative (a.e. stands for "almost everywhere"), but, on the other hand, as F_w is only "almost everywhere" determined by μ_w, one can assume with no loss of generality that $F_w \geqslant 0$. Moreover, $F_w \in L^1(\Gamma)$, since μ_w is finite.

Note that for fixed $(q, p) \in \Gamma$, the correspondence

$$w \mapsto F_w(q, p) \tag{15}$$

after extending F: $w \mapsto F_w$ to a contracting positive linear map $\tilde{F}$: $B^1(H) \to L^1(\Gamma)$, defines a positive linear functional

$$v \mapsto [\tilde{F}(v)](q, p) \tag{16}$$

on the space $B^1(H)$, and therefore continuous. Thus, there exists a bounded positive operator $\hat{F}(q, p)$ on H such that

$$\mathrm{Tr}\big(\hat{F}(q, p)v\big) = \big[\tilde{F}(v)\big](q, p) \tag{17}$$

for all v in $B^1(H)$. In particular, for w in S

$$\mathrm{Tr}\big(\hat{F}(q, p)w\big) = F_w(q, p) \tag{18}$$

We shall say that an absolutely continuous phase-space representation F: $w \mapsto F_w$ $(w \in S)$ is *continuous* (respectively, *bounded*) if the map $\hat{F}$: $(q, p) \mapsto \hat{F}(q, p)$ is continuous (respectively, bounded).

Note that for a fixed (q, p) in Γ, the map

$$w \mapsto F_w(q, p), \qquad w \in S$$

where F is an arbitrary absolutely continuous phase-space representation, is clearly bounded, since

$$\begin{aligned} F_w(q, p) &= \mathrm{Tr}\big(\hat{F}(q, p)w\big) = \|\hat{F}(q, p)w\|_1 \\ &\leqslant \|\hat{F}(q, p)\|\|w\|_1 = \|\hat{F}(q, p)\| \end{aligned}$$

and therefore

$$\sup\{F_w(q, p): w \in S\} \leqslant \|\hat{F}(q, p)\| \tag{19}$$

It is not difficult to prove that F: $w \mapsto F_w$ is continuous if and only if the family $\{F_w: w \in S\}$, representing the image of the set S of quantum states under the mapping F, is *equicontinuous* in the sense that

$$\begin{aligned} &\forall_{(q,p) \in \Gamma} \forall_{\varepsilon > 0} \exists_{\delta > 0} \forall_{(q', p')} |(q, p) - (q', p')| < \delta \\ &\qquad \Rightarrow \forall_{w \in S} |F_w(q, p) - F_w(q', p')| < \varepsilon \end{aligned} \tag{20}$$

Indeed, for an arbitrary $w \in S$

$$\begin{aligned} |F_w(q, p) - F_w(q', p')| &= |\mathrm{Tr}\big(\hat{F}(q, p)w\big) - \mathrm{Tr}\big(\hat{F}(q', p')w\big)| \\ &= \big|\|\hat{F}(q, p)w\|_1 - \|\hat{F}(q', p')w\|_1\big| \leqslant \|\hat{F}(q, p)w - \hat{F}(q', p')w\|_1 \\ &\leqslant \|\hat{F}(q, p) - \hat{F}(q', p')\|\|w\|_1 = \|\hat{F}(q, p) - \hat{F}(q', p')\| \end{aligned}$$

Hence, the continuity of $\hat{F}$ implies clearly the equicontinuity of $\{F_w\}$.

Conversely,

$$\begin{aligned} \|\hat{F}(q, p) - \hat{F}(q', p')\| &= \sup_{\|\psi\| = 1} |\big(\psi, [\hat{F}(q, p) - \hat{F}(q', p')]\psi\big)| \\ &= \sup_{\|\psi\| = 1} |\mathrm{Tr}\big([\hat{F}(q, p) - \hat{F}(q', p')]P_\psi\big)| \end{aligned}$$

where P_ψ denotes the orthogonal projection onto the one-dimensional subspace of H spanned by $\psi \in H$, so we have

$$\|\hat{F}(q,p)-\hat{F}(q',p')\| = \sup_{\|\psi\|=1} |F_\psi(q,p)-F_\psi(q',p')| \tag{21}$$

where F_ψ stands for F_w with $w=P_\psi$, and by (21) we see that the equicontinuity of $\{F_w\}$ implies the continuity of $\hat{F}$, which completes the proof of our statement.

Similarly, F: $w \mapsto F_w$ is bounded if and only if the family $\{F_w: w \in S\}$ is *equibounded*, i.e.,

$$\exists_{K>0} \forall_{(q,p)\in\Gamma} \forall_{w\in S} F_w(q,p) \leqslant K \tag{22}$$

The proof is essentially the same as the preceding one. For an arbitrary $w \in S$

$$F_w(q,p) = \|\hat{F}(q,p)w\|_1 \leqslant \|\hat{F}(q,p)\|$$

and hence it follows that the boundedness of $\hat{F}$ implies that $\{F_w\}$ is equibounded. Conversely,

$$\|\hat{F}(q,p)\| = \sup_{\|\psi\|=1} (\psi, \hat{F}(q,p)\psi) = \sup_{\|\psi\|=1} \mathrm{Tr}\big(\hat{F}(q,p)P_\psi\big)$$

$$= \sup_{\|\psi\|=1} F_\psi(q,p)$$

so that the equiboundedness of $\{F_w: w \in S\}$ implies that $\hat{F}$ is bounded. Note finally that *the family of operators* $\hat{F}(q,p)$ *provide a continuous resolution of the identity* in the sense that

$$\int_\Gamma \hat{F}(q,p)\, dq\, dp = I \tag{23}$$

where I stands for the identity operator on H.

The integral above is clearly meant in the weak sense, that is, for every $w \in S$

$$\int_\Gamma \mathrm{Tr}\big(\hat{F}(q,p)w\big)\, dq\, dp = 1 \tag{24}$$

[which is a direct consequence of (18)] or, equivalently,

$$\int_\Gamma (\psi, \hat{F}(q,p)\psi)\, dq\, dp = \|\psi\|^2 \tag{25}$$

for all $\psi \in H$.

Inserting (18) into the classical mean value $\langle f; F_w \rangle_c$ we find

$$\begin{aligned}\langle f; F_w \rangle_c &= \int_\Gamma f(q,p) F_w(q,p)\, dq\, dp \\ &= \int_\Gamma f(q,p) \mathrm{Tr}(\hat{F}(q,p) w)\, dq\, dp \\ &= \mathrm{Tr}\left[w \int_\Gamma f(q,p) \hat{F}(q,p)\, dq\, dp \right]\end{aligned}$$

so that we get by (3)

$$A(f) = \int_\Gamma f(q,p) \hat{F}(q,p)\, dq\, dp \tag{26}$$

where, as before, the integral above is clearly a weak integral. Hence, in particular, for f identically equal to 1 we get

$$A(1) = \int_\Gamma \hat{F}(q,p)\, dq\, dp = I \tag{27}$$

To summarize, one can say that *every absolutely continuous phase-space representation map F: $w \mapsto F_w$ can alternatively be described by its associated continuous resolution of the identity $\hat{F}(q,p)$, each $\hat{F}(q,p)$ being a positive bounded operator acting in the Hilbert space H corresponding to a given quantum system.* The original phase-space representation mapping F can then be recovered according to the prescription (18):

$$F_w(q,p) = \mathrm{Tr}(\hat{F}(q,p) w), \qquad w \in S \tag{28}$$

We shall now pass on to the case of a *bounded* absolutely continuous phase space representation F: $w \mapsto F_w(q,p)$, $w \in S$, i.e., we assume that the corresponding identity resolution $\hat{F}$: $(q,p) \mapsto \hat{F}(q,p)$ is a bounded function:

$$\exists_{K>0} \forall_{(q,p)\in\Gamma} \|\hat{F}(q,p)\| \leqslant K \tag{29}$$

The assumption above implies that the probability density (28) is sufficiently well behaved (for any w) in the sense that its *characteristic function*

$$M_w(x, y) = \int_\Gamma F_w(q, p) e^{i(x\cdot q + y\cdot p)}\, dq\, dp \tag{30}$$

[which is the mean value of the function $(q, p) \mapsto e^{i(x\cdot q+y\cdot p)}$ in the "classical state" F_w] contains as much information as the probability density F_w itself, since (30) can then be inverted to yield

$$\begin{aligned} F_w(q, p) &= (2\pi)^{-2n} \int_\Gamma M_w(x, y) e^{-i(x\cdot q+y\cdot p)}\, dx\, dy \\ &= (2\pi)^{-2n} \int_\Gamma \mathrm{Tr}\big[A(e^{i(x\cdot + y\cdot)})w\big] e^{-i(x\cdot q+y\cdot p)}\, dx\, dy \end{aligned}$$

where the last equality is derived with the help of (3) ($e^{i(x\cdot+y\cdot)}$ denotes here the function $(q, p) \mapsto e^{i(x\cdot q+y\cdot p)}$), and hence

$$\hat{F}(q, p) = (2\pi)^{-2n} \int_\Gamma A(e^{i(x\cdot+y\cdot)}) e^{-i(x\cdot q+y\cdot p)}\, dx\, dy \tag{31}$$

Indeed, by using (28) and (29) we easily find that the probability density F_w is a bounded function:

$$\begin{aligned} 0 &\leqslant F_w(q, p) = \mathrm{Tr}(\hat{F}(q, p)w) = \|\hat{F}(q, p)w\|_1 \\ &\leqslant \|\hat{F}(q, p)\|\|w\|_1 = \|\hat{F}(q, p)\| \leqslant K \end{aligned}$$

for all (q, p) in Γ, so that F_w, being a member of $L^1(\Gamma)$, must also belong to $L^2(\Gamma)$, and therefore M_w in (30) is well defined as the Fourier–Plancherel transform of $(2\pi)^n F_w$.

We thus see from (31) that in the case of a bounded completely continuous phase-space representation F: $w \mapsto F_w$, the associated identity resolution $\hat{F}(q, p)$ is completely determined by the family of operators $A(e^{i(x\cdot+y\cdot)})$, which is in fact a representation of the dual group of $\Gamma = \mathbb{R}^{2n}$ in $B(H)$, and conversely, $\hat{F}(q, p)$ determines $A(e^{i(x\cdot+y\cdot)})$ according to (26).

In other words, any *bounded completely continuous phase-space representation F can equivalently be specified by associating a (bounded) operator $A(e^{i(x\cdot+y\cdot)})$ with the exponential function $e^{i(x\cdot+y\cdot)}$*, in accordance with the observation made by Weyl many years ago (Weyl, 1950). His famous "rule of association" was

$$e^{i(x\cdot+y\cdot)} \mapsto e^{i(x\cdot Q+y\cdot P)^{\sim}} \equiv W(y, x) \tag{32}$$

where denotes the closure of an operator (see, e.g., Akhiezer and Glazman, 1981, Vol. 1, p. 121) and Q, P denote the n-tuples of, respectively, position and momentum operators in H satisfying the canonical commutation relations $[Q_i, P_j] = i\hbar\delta_{ij}I$, which imply the following commutation relations for the *Weyl algebra* $\{W(x, y)\colon (x, y) \in \mathbb{R}^{2n}\}$:

$$W(x,y)W(x',y') = \exp\left[\frac{i\hbar}{2}(x\cdot y' - x'\cdot y)\right]W(x+x', y+y') \quad (33)$$

A large class of generalized "association rules" has been considered by Cohen (1966a), Agarwal and Wolf (1970), and Srinivas and Wolf (1975), and all the well-known correspondences are particular cases of the so-called "*Ω-rules of association*" (see Srinivas and Wolf, 1975) which are of the form

$$e^{i(x\cdot + y\cdot)} \mapsto \Omega(x,y)\, e^{i(x\cdot Q + y\cdot P)^{\sim}} \quad (34)$$

where Ω is assumed to be sufficiently well behaved [usually it is assumed (Agarwal and Wolf, 1970; Srinivas and Wolf, 1975) that $\Omega(x, y)$ is the boundary value of an entire analytic function in complex variables, and has no zeros for real x, y] and has the further properties that

$$\Omega(0,0) = 1$$

and

$$\bar{\Omega}(x,y) = \Omega(-x,-y)$$

where the bar denotes complex conjugation.

Below, we list some of the well-known rules of association which were discussed in great detail by Agarwal and Wolf (1970):

Ω	Rule of association
1	Weyl–Wigner–Moyal
$\cos(x\cdot y/2)$	Symmetric
$\exp[x^2 + y^2)/4]$	Normal
$\exp[-(x^2 + y^2)/4]$	Antinormal

This table should be completed by a one more example, which is a direct generalization of Weyl's association rule:

$$\Omega(x,y) = \exp\left\{-\tfrac{1}{2}\left[(a\cdot x)^2 + (b\cdot y)^2\right]\right\} \quad (35)$$

where a, b are two vectors in $\mathbb{R}^n$ with constant nonnegative coordinates. Clearly, (35) is an immediate generalization of the Weyl's association rule W, as the latter corresponds to the choice $a = b = 0$. Moreover, according to (31)

$$\hat{F}(q, p) = (2\pi)^{-2n} \int_\Gamma e^{-i(x\cdot q + y\cdot p)} A(e^{i(x\cdot + y\cdot)})\, dx\, dy$$

$$= (2\pi)^{-2n} \int_\Gamma e^{-i(x\cdot q + y\cdot p)} e^{-[(a\cdot x)^2 + (b\cdot y)^2]/2} W(y, x)\, dx\, dy$$

so that $\hat{F}(q, p)$ is proportional to the projection operator T_{qp} extensively studied by Schroeck (1981, 1982):

$$\hat{F}(q, p) = (2\pi\hbar)^{-n} T_{qp} \tag{36}$$

If one puts

$$a_i = \operatorname{Var} Q_i, \qquad b_i = \operatorname{Var} P_i, \qquad i = 1, 2, 3, \ldots, n$$

then T_{qp} projects onto the minimum uncertainty state with $\exp Q_i = q_i$ and $\exp P_i = p_i$ (see Schroeck, 1981).

From now on, we shall assume, for simplicity, that

$$\hat{F}(q, p) = cP_{qp}$$

where P_{qp} is an orthogonal projector, and c is a positive real constant independent of the phase-space point (q, p). We therefore have a resolution of the identity

$$\int_\Gamma P_{qp}\, dq\, dp = c^{-1} I$$

where the integral above is meant, as before, in the weak sense.

The identity resolution $e = \{P_{qp}\}$ is said to be *atomic* if each of the projections P_{qp} projects onto a one-dimensional subspace. Otherwise, we call e *nonatomic*.

3. SOME PHYSICAL EXAMPLES

The most important case of an atomic resolution of the identity

$$\int_\Gamma P_{qp}\, dq\, dp = c^{-1} I, \qquad c > 0 \tag{37}$$

is obtained when $\{P_{qp}\}$ comes from a fixed (but arbitrary) vector ψ in H, according to the following prescription (Ali and Prugovečki, 1977):

P_{qp} = the orthoprojector onto the one-dimensional subspace of H spanned by $U(q, p)\psi$

where $U(q, p)$ is a projective (or ray) representation of the additive group of $\mathbb{R}^{2n}$.

Example 1. Let G be the proper Galilei group specified by the following multiplication rule for an arbitrary pair $g_1 = (b_1, \mathbf{a}_1, \mathbf{v}_1, R_1)$, $g_2 = (b_2, \mathbf{a}_2, \mathbf{v}_2, R_2)$ of its elements (see, for instance, Lévy-Leblond, 1971):

$$g_1 g_2 = (b_1 + b_2, \mathbf{a}_1 + R_1\mathbf{a}_2 + b_2\mathbf{v}_1, \mathbf{v}_1 + R_1\mathbf{v}_2, R_1 R_2)$$

We recall that the action of proper Galilei transformations $g = (b, \mathbf{a}, \mathbf{v}, R)$ on $\Gamma \oplus \mathbb{R}^1$ is as follows (Lévy-Leblond, 1971):

$$(\mathbf{q}, \mathbf{p}, t) \mapsto (\mathbf{q}', \mathbf{p}', t') \equiv g \circ (\mathbf{q}, \mathbf{p}, t) = (R\mathbf{q} + \mathbf{v}t + \mathbf{a}, R\mathbf{p} + m\mathbf{v}, t + b)$$

Let $g \mapsto U(g)$ be a projective representation of G in the Hilbert space of square-integrable complex-valued functions $\psi(\mathbf{q}, \mathbf{p}, t)$ on $\Gamma \oplus \mathbb{R}^1$ defined by (Prugovečki, 1978):

$$[U(b, \mathbf{a}, \mathbf{v}, R)\psi](\mathbf{q}, \mathbf{p}, t) = \exp\left[\frac{i}{\hbar}\left(-\frac{m\mathbf{v}^2}{2}(t - b) + m\mathbf{v}\cdot(\mathbf{q} - \mathbf{a})\right)\right]$$
$$\times \psi\left(R^{-1}(\mathbf{q} - \mathbf{v}(t - b) - \mathbf{a}), R^{-1}(\mathbf{p} - m\mathbf{v}), t - b\right). \tag{38}$$

Consequently, the law of ray representation multiplication for U is (Prugovečki, 1978)

$$U(b_1, \mathbf{a}_1, \mathbf{v}_1, R_1)U(b_2, \mathbf{a}_2, \mathbf{v}_2, R_2) = \exp\left[\frac{i}{\hbar}\left(\frac{m\mathbf{v}_1^2}{2}b_2 + m\mathbf{v}_1\cdot R_1\mathbf{a}_2\right)\right]$$
$$\times U(b_1 + b_2, \mathbf{a}_1 + R_1\mathbf{a}_2 + b_2\mathbf{v}_1, \mathbf{v}_1 + R_1\mathbf{v}_2, R_1 R_2) \tag{39}$$

If we restrict ourselves to the case of fixed t, say $t = 0$, then we are naturally led to consider the U-subrepresentation of the subgroup

$$G_0 = \{g \in G : g = (0, \mathbf{a}, \mathbf{v}, \mathbb{1})\} \tag{40}$$

in the Hilbert space of square-integrable complex-valued functions $\psi(\mathbf{q}, \mathbf{p})$

on the phase space Γ, which is, according to (38), of the form

$$\left[U\left(0,\mathbf{q},\frac{\mathbf{p}}{m},\mathbb{1}\right)\psi\right](\mathbf{q}',\mathbf{p}')=\exp\left[\frac{i}{\hbar}\mathbf{p}\cdot(\mathbf{q}'-\mathbf{q})\right]\psi(\mathbf{q}'-\mathbf{q},\mathbf{p}'-\mathbf{p}) \tag{41}$$

and it is not difficult to check that the family of orthogonal projections $P_{\mathbf{qp}}$ onto the one-dimensional subspaces of $L^2(\Gamma)$ spanned by $\psi_{\mathbf{qp}} = U(0,\mathbf{q},\mathbf{p}/m,\mathbb{1})\psi$, ψ being a fixed norm one vector in $L^2(\Gamma)$, obeys (37) with $c=(2\pi\hbar)^{-3}$.

Example 2. By considering the conventional unitary ray representation $U(b,\mathbf{a},\mathbf{v},R)$ of the proper Galilei group G on the Hilbert space $L^2(\mathbb{R}^3)$, we obtain for the corresponding U-subrepresentation $U(0,\mathbf{q},\mathbf{p}/m,\mathbb{1})$ of the subgroup G_0 [compare (41)]

$$\left[U\left(0,\mathbf{q},\frac{\mathbf{p}}{m},\mathbb{1}\right)\psi\right](\mathbf{x})=\exp\left[\frac{i}{\hbar}\mathbf{p}\cdot(\mathbf{x}-\mathbf{q})\right]\psi(\mathbf{x}-\mathbf{q}) \tag{42}$$

where $\psi\in L^2(\mathbb{R}^3)$, and the family $\psi_{\mathbf{qp}}=U(0,\mathbf{q},\mathbf{p}/m,\mathbb{1})\psi$, ψ being a fixed member of $L^2(\mathbb{R}^3)$ with norm one, is again easily proven (Ali and Prugovečki, 1977) to provide a continuous resolution of the identity $\{P_{\mathbf{qp}}\}$ on $L^2(\mathbb{R}^3)$.

Example 3. Let H be the Hilbert space of square-integrable complex-valued functions on $\mathbb{R}^n$, and let $W(q,p)$ be a strongly continuous representation of the additive group of $\mathbb{R}^{2n}$ into the *Weyl algebra* on H, that is, $W(q,p)$ are unitary operators on H satisfying the commutation relations

$$W(q,p)W(q',p')=\exp\left[\frac{i\hbar}{2}(q\cdot p'-q'\cdot p)\right]W(q+q',p+p') \tag{43}$$

It can then be shown that the family of vectors $\psi_{qp}=W(q,p)\psi$, where ψ is a fixed element of H with unit norm, provide a continuous resolution of the identity in the sense that the corresponding family P_{qp} of one-dimensional projectors obeys (37) with $c=(\hbar/2\pi)^n$. Moreover, it is interesting to note that the above Weyl algebra representation is equivalent to that of Example 2 (for $n=3$). Indeed, we have

$$[W(a\mathbf{q},b\mathbf{p})\psi](\mathbf{x})=\exp\left[ib\mathbf{p}\cdot\left(\mathbf{x}+\frac{\hbar}{2}a\mathbf{q}\right)\right]\psi(\mathbf{x}+\hbar a\mathbf{q}),\qquad a,b\in\mathbb{R}^1$$

so that for $a=-\hbar^{-1}$ and $b=\hbar^{-1}$ one obtains

$$\begin{aligned}[W(-\hbar^{-1}\mathbf{q},\hbar^{-1}\mathbf{p})\psi](\mathbf{x})&=\exp\left[\frac{i}{\hbar}\mathbf{p}\cdot(\mathbf{x}-\tfrac{1}{2}\mathbf{q})\right]\psi(\mathbf{x}-\mathbf{q})\\&=\exp\left[\frac{i}{2\hbar}\mathbf{p}\cdot\mathbf{q}\right]\left\{U\left(0,\mathbf{q},\frac{\mathbf{p}}{m},\mathbb{1}\right)\psi\right\}(\mathbf{x})\end{aligned}$$

that is,

$$U\left(0, \mathbf{q}, \frac{\mathbf{p}}{m}, \mathbb{1}\right) = \exp\left[-\frac{i}{2\hbar}\mathbf{p}\cdot\mathbf{q}\right] W(-\hbar^{-1}\mathbf{q}, \hbar^{-1}\mathbf{p}) \tag{44}$$

Example 4. Let $\Gamma = \mathbb{R}^{2n}$, $H = L^2(\mathbb{R}^n)$, and

$$U(q, p) = V(\hbar^{-1}p)U(-\hbar^{-1}q) \tag{45}$$

with $V(p) = \exp(ip\cdot Q)$, $U(q) = \exp(iq\cdot P)$, where Q and P denote the n-tuples of, respectively, position and momentum operators in H. Recall that $[U(q)\psi](x) = \psi(x + \hbar q)$, $[V(p)\psi](x) = \exp(ip\cdot x)\psi(x)$, where $\psi \in L^2(\mathbb{R}^n)$.

We clearly have

$$U(q, p) = \exp\left(\frac{i}{2\hbar}q\cdot p\right) W(-\hbar^{-1}q, \hbar^{-1}p) \tag{46}$$

so that

$$[U(q, p)\psi](x) = \exp\left(\frac{i}{\hbar}p\cdot x\right)\psi(x - q), \qquad \psi \in L^2(\mathbb{R}^n) \tag{47}$$

It has been shown (Ali and Prugovečki, 1977) that the family $\{P_{qp}\}$ of orthoprojectors onto the one-dimensional subspaces of H spanned by $U(q, p)\psi$, where ψ is a fixed element of $L^2(\mathbb{R}^n)$ with norm one, satisfies (37) with $c = (2\pi\hbar)^{-n}$.

4. THE CASE OF AN ATOMIC IDENTITY RESOLUTION

We shall now return to the general scheme of a phase-space representation of the Hilbert-space quantum mechanics described in Section 2, and concentrate our attention on the particular (but evidently the most important) case of an *atomic* identity resolution

$$\int_\Gamma P_{qp}\, dq\, dp = c^{-1}I, \qquad c > 0 \tag{48}$$

resulting from an overcomplete family of vectors

$$\psi_{qp} = U(q, p)\psi \tag{49}$$

where ψ is a fixed (but arbitrary) nonzero vector in a Hilbert space H, and

U stands for a (fixed) map from $\Gamma = \mathbb{R}^{2n}$ into $B(H)$, the algebra of bounded operators on H. We recall that P_{qp} is defined as the orthogonal projector onto the one-dimensional subspace of H spanned by ψ_{qp}.

Our main aim in this section is to give several equivalent descriptions of the concept of this particular case of a phase-space representation of quantum mechanics.

Let us define the mapping

$$M\colon (\psi, \phi) \mapsto M[\psi, \phi] \tag{50}$$

from $H \times H$ into $L^2(\Gamma)$ by setting

$$M[\psi, \phi](q, p) = c^{1/2}(U(q, p)\phi, \psi) \tag{51}$$

It is easy to verify that M is a sesquilinear map of $H \times H$ into $L^2(\Gamma)$ such that for all ψ and ϕ in H

$$\|M[\psi, \phi]\|_{L^2(\Gamma)} = \|\psi\|\|\phi\| \tag{52}$$

The equality above leads (and is actually equivalent) to

$$(M[\psi, \phi], M[\psi', \phi'])_{L^2(\Gamma)} = (\psi, \psi')(\phi', \phi) \tag{53}$$

so that we arrive at a striking similarity with the formula

$$(\psi \otimes \phi, \psi' \otimes \phi')_2 = (\psi, \psi')(\phi', \phi) \tag{54}$$

where $(\cdot, \cdot)_2$ stands for the Hilbert–Schmidt scalar product, and $\psi \otimes \phi$ is the bounded operator on H ($\psi, \phi \in H$) defined by (see, e.g., Ringrose, 1971; or Schatten, 1960)

$$(\psi \otimes \phi)\xi = (\phi, \xi)\psi, \qquad \xi \in H \tag{55}$$

One can therefore expect to be able to establish an isometry from $B^2(H)$, the Hilbert space of Hilbert–Schmidt operators on H, into $L^2(\Gamma)$, and this will indeed be shown later on.

Remark. The isometric embedding

$$\psi \mapsto M_e\psi \qquad (e, \psi \in H, e \text{ fixed})$$

of the Hilbert space H into $L^2(\Gamma)$, considered by Ali and Prugovečki (1977), is obviously related to the mapping M as follows:

$$M_e\psi = \|e\|^{-1}M[\psi, e] \tag{56}$$

Now, by using the Riesz representation theorem we can prove the following result:

There exists a unique bounded linear map

$$V\colon f \mapsto V(f)$$

of $L^2(\Gamma)$ *into* $B(H)$, *the algebra of bounded linear operators on* H, *such that*

$$(\psi, V(f)\phi) = \int_\Gamma f(q,p)\,\overline{M[\psi,\phi]}\,(q,p)\,dq\,dp \tag{57}$$

for all $\psi, \phi \in H$ *and all* $f \in L^2(\Gamma)$.

Proof. For a fixed f in $L^2(\Gamma)$ we define a bounded sesquilinear form a_f on H by

$$a_f(\psi,\phi) = \int_\Gamma f(q,p)\,\overline{M[\psi,\phi]}\,(q,p)\,dq\,dp$$

$$= (M[\psi,\phi], f)_{L^2(\Gamma)}$$

Note that the boundedness of a_f is almost obvious, since for all ψ and ϕ in H

$$|a_f(\psi,\phi)| \leqslant |(M[\psi,\phi], f)_{L^2(\Gamma)}|$$

$$\leqslant \|M[\psi,\phi]\|_{L^2(\Gamma)} \|f\|_{L^2(\Gamma)}$$

$$= \|\psi\| \|\phi\| \|f\|$$

where the last equality was derived with the help of (52).

Therefore, there exists by Riesz representation theorem a unique bounded operator $V(f)$ on H such that

$$a_f(\psi,\phi) = (\psi, V(f)\phi)$$

for all $\psi, \phi \in H$, and the rest of the theorem follows in a straightforward manner.

Now, the following statement relates the linear map $f \mapsto V(f)$ introduced above to the Ali–Prugovečki correspondence $\psi \mapsto M_e\psi$:

For a fixed norm one vector e in H, *the map*

$$f \mapsto V(f)e$$

of $L^2(\Gamma)$ into H, when restricted to the subspace $L^2(\Gamma_e) \equiv M_e(H)$, is the inverse of the Ali–Prugovečki embedding $\psi \mapsto M_e\psi$, $\psi \in H$.

Proof. By the preceding theorem and by (56) we have (ϕ, ψ, d, e are here arbitrary elements of H):

$$\begin{aligned}(\phi, V(M_d\psi)e) &= \int_\Gamma (M_d\psi)(q,p)\overline{M[\phi,e]}(q,p)\,dq\,dp \\ &= \|d\|^{-1}\int_\Gamma M[\psi,d](q,p)M[\phi,e](q,p)\,dq\,dp \\ &= \|d\|^{-1}(M[\phi,e], M[\psi,d])_{L^2(\Gamma)} \\ &= \|d\|^{-1}(\phi,\psi)(d,e)\end{aligned}$$

where the last equality is obtained by using (53).

The equality above, valid for all ϕ in H, gives us

$$V(M_d\psi)e = \|d\|^{-1}(d,e)\psi \tag{58}$$

hence, in particular,

$$V(M_e\psi)e = \psi$$

for all $\psi \in H$, which concludes the proof of the theorem.

Note that by using (55) and (56) one can rewrite (58) in the form

$$(\psi \otimes d)e = (d,e)\psi = \|d\|V(M_d\psi)e = V(M[\psi,d])e$$

ψ, d, e being arbitrary elements of H. Hence

$$V(M[\psi,d]) = \psi \otimes d \tag{59}$$

for any pair ψ, d of vectors in H, and clearly [see (54) and (52)]

$$\|V(M[\psi,d])\|_2 = \|M[\psi,d]\|_{L^2(\Gamma)} \tag{60}$$

where $\|\cdot\|_2$ stands for the Hilbert–Schmidt norm (for a definition, see, e.g., Schatten, 1960; or Ringrose, 1971).

Let now $L_0^2(\Gamma)$ denote the closed subspace of $L^2(\Gamma)$ spanned by the functions $M[\psi,\phi]$, where ψ and ϕ run over the Hilbert space H. It is almost obvious that the map $f \mapsto V(f)$, $f \in L^2(\Gamma)$, when restricted to $L_0^2(\Gamma)$, is a unitary mapping onto $B^2(H)$, the Hilbert space of the Hilbert–Schmidt operators on H. Indeed, the map V, when restricted to the linear span L of the set $\{M[\psi, \phi]: \psi, \phi \in H\}$, is obviously an isometry, and, clearly, $V(L)$ is identical with the set of the operators of finite rank on H. Since the later form a dense set (with respect to the Hilbert–Schmidt norm $\|\cdot\|_2$) in $B^2(H)$ (see, e.g., Schatten, 1960), we see that $V|_{L_0^2(\Gamma)}$ must be an isometry *onto* $B^2(H)$, as claimed.

We therefore have an isometry

$$U: B^2(H) \to L^2(\Gamma)$$

$$U = V|_{L_0^2(\Gamma)}^{-1} \tag{61}$$

as claimed before.

Remark. The above-mentioned isometry is clearly a *-monomorphism of $B^2(H)$, considered as an H^*-algebra, into the H^*-algebra $L^2(\Gamma)$ (for a definition of an H^*-algebra, see, for example, Rickart, 1960).

Moreover, since as an H^*-algebra, $L^2(\Gamma) = L^2(\mathbb{R}^n \times \mathbb{R}^n)$ is *-isomorphic to the H^*-algebra $B^2(L^2(\mathbb{R}^n))$ of the Hilbert–Schmidt operators on $L^2(\mathbb{R}^n)$ (see, e.g., Ringrose, 1971), we actually have a *-monomorphism of the Hilbert–Schmidt H^*-algebras associated with H and $L^2(\mathbb{R}^n)$, respectively:

$$B^2(H) \subsetneq B^2(L^2(\mathbb{R}^n)) \tag{62}$$

To conclude our discussion of the concept of an atomic identity resolution $\{P_{qp}\}$ resulting from an overcomplete family of vectors $\psi_{qp} = U(q \cdot p)\psi$, $\psi \in H$, with U denoting a (fixed) map from Γ into $B(H)$, we shall summarize the results we have obtained in this section in the following single theorem:

Theorem. The following two statements are equivalent:

(1) There is a map U: $(q, p) \mapsto U(q, p)$ of the phase space $\Gamma = \mathbb{R}^{2n}$ into $B(H)$, the algebra of bounded operators on the Hilbert space H, such that for all ϕ and ψ in H

$$\int_\Gamma (\phi, U(q, p)\psi)(U(q, p)\psi, \phi)\, dq\, dp = c^{-1}\|\phi\|^2\|\psi\|^2$$

where c is a positive real constant.

(2) There exists a sesquilinear map

$$M: (\phi, \psi) \mapsto M[\phi, \psi]$$

from $H \times H$ into $L^2(\Gamma)$ such that

(i) For all ϕ and ψ in H

$$\|M[\phi,\psi]\|_{L^2(\Gamma)} = \|\phi\|\|\psi\|$$

(ii) For a fixed pair q, p in Γ the sesquilinear form on H defined by

$$(\phi,\psi) \mapsto M[\phi,\psi](q,p)$$

is bounded.

Moreover, the first half of the statement (2) is equivalent to the following assertion:

(3) There is a *-monomorphism of the H^*-algebra $B^2(H)$ of the Hilbert–Schmidt operators on H into $L^2(\Gamma)$, the H^*-algebra of square-integrable complex-valued functions on the phase space Γ.

Proof. Until now we have shown the implications

$$(1) \Rightarrow (2i) \Rightarrow (3)$$

The implication from (3) to (2i) is straightforward, so there remain to be shown the implications (1) ⇒ (2ii) and (2) ⇒ (1). The first one is trivial, since by (51) we get

$$\begin{aligned}|M[\phi,\psi](q,p)| &= c^{1/2}|(U(q,p)\psi,\phi)| \\ &\leqslant c^{1/2}\|U(q,p)\|\|\psi\|\|\phi\|\end{aligned}$$

so that the boundedness of $M[\phi,\psi](q,p)$ follows as an immediate consequence of the boundedness of the operator $U(q,p)$.

We shall now prove the implication from (2) to (1). By (2ii), for a fixed pair q, p in Γ the sesquilinear form

$$(\phi,\psi) \mapsto M[\phi,\psi](q,p)$$

is bounded, and hence there exists by Riesz representation theorem a (unique) bounded linear operator $U(q,p)$ on H such that

$$(\phi, U(q,p)\psi) = c^{-1/2}\,\overline{M[\phi,\psi]}\,(q,p)$$

where the bar denotes, as usually, the complex conjugation. Now, by using (2i) one gets for all $\phi, \psi \in H$

$$\begin{aligned}\int_\Gamma (\phi, U(q,p)\psi)(U(q,p)\psi,\phi)\,dq\,dp &= c^{-1}(M[\phi,\psi], M[\phi,\psi])_{L^2(\Gamma)} \\ &= c^{-1}\|\phi\|^2\|\psi\|^2\end{aligned}$$

The statement (1) is therefore proven, which concludes the proof of the theorem.

5. PHASE SPACE REPRESENTATIONS OF THE QUANTUM-MECHANICAL HILBERT SPACE

We shall begin with the more general case of a nonatomic identity resolution $e = \{P_{qp}\colon (q, p) \in \Gamma\}$, where P_{qp} is a family of orthogonal projections onto the closed subspaces of H satisfying

$$\int_\Gamma P_{qp}\, dq\, dp = c^{-1} I, \qquad c > 0 \tag{63}$$

and consider the passage to a phase-space representation of the quantum-mechanical Hilbert space H given by the correspondence

$$\psi \to \hat{M}_e \psi \in L^2(\Gamma, H), \qquad \psi \in H \tag{64}$$

where $L^2(\Gamma, H)$ denotes the Hilbert space of Lebesgue square-integrable functions from Γ into H (see, e.g., Yosida, 1968), and $\hat{M}_e\psi$ is defined by

$$(\hat{M}_e\psi)(q, p) = c^{1/2} P_{qp}\psi, \qquad \psi \in H \tag{65}$$

It is clear that $\hat{M}_e\psi$ belongs to $L^2(\Gamma, H)$, and furthermore

$$\|\hat{M}_e\psi\|^2_{L^2(\Gamma, H)} = \int_\Gamma \big((\hat{M}_e\psi)(q, p), (\hat{M}_e\psi)(q, p)\big)\, dq\, dp$$

$$= c\int_\Gamma (\psi, P_{qp}\psi)\, dq\, dp = \|\psi\|^2$$

which shows that $\hat{M}_e\colon H \to L^2(\Gamma, H)$ *is an isometry* (see also Schroeck, 1982).

Let now $\mathbb{P}_e$ denote the orthogonal projection of $L^2(\Gamma, H)$ onto $\hat{M}_e(H)$. Repeating the arguments of Ali and Prugovečki (1977b) we may show that $\mathbb{P}_e$ can be represented by an integral operator on $L^2(\Gamma, H)$; namely,

$$(\mathbb{P}_e\Phi)(q, p) = \int_\Gamma \hat{K}_e(q, p; q', p')\Phi(q', p')\, dq'\, dp', \qquad \Phi \in L^2(\Gamma, H) \tag{66}$$

with the operator-valued kernel

$$\hat{K}_e(q, p; q', p') = cP_{qp}P_{q'p'} \tag{67}$$

Indeed, it can be easily shown that

$$\hat{K}_e(q, p; q', p') = \int_\Gamma \hat{K}_e(q, p; q'', p'')\hat{K}_e(q'', p''; q', p')\, dq''\, dp''$$

so that $\hat{K}_e$ is a *reproducing kernel* in $L^2(\Gamma, H)$, which implies that $\mathbb{P}_e$, defined by (66), is an idempotent. We furthermore have

$$\hat{K}_e^*(q, p; q', p') = \hat{K}_e(q', p'; q, p) \tag{68}$$

* denoting the Hermitian conjugation, which implies that $\mathbb{P}_e$ is self-adjoint.

Moreover, $\mathbb{P}_e$ leaves Φ unchanged if $\Phi \in \hat{M}_e(H)$, which means that $\hat{M}_e(H) \subseteq \mathbb{P}_e(L^2(\Gamma, H))$, and we actually have

$$\hat{M}_e(H) = \mathbb{P}_e(L^2(\Gamma, H)) \tag{69}$$

since $\mathbb{P}_e\Psi = 0$ whenever $\Psi \in \hat{M}_e(H)^\perp$.

Note finally that the correspondence

$$e \mapsto \mathbb{P}_e \tag{70}$$

is one-to-one.

Indeed, if we have $\mathbb{P}_e = \mathbb{P}_f$, where $e = \{P_{qp}\}$ and $f = \{Q_{qp}\}$ are two identity resolutions obeying (63), then the equality

$$(\mathbb{P}_e\Phi)(q, p) = (\mathbb{P}_f\Phi)(q, p)$$

valid for all $q, p \in \Gamma$ and all Φ in $L^2(\Gamma, H)$, and being equivalent to

$$\int_\Gamma P_{qp}P_{q'p'}\Phi(q', p')\, dq'\, dp' = \int_\Gamma Q_{qp}Q_{q'p'}\Phi(q', p')\, dq'\, dp'$$

leads to

$$P_{qp}P_{q'p'} = Q_{qp}Q_{q'p'}$$

for all (q, p) and (q', p') in Γ, and hence, after substituting $q' = q$, $p' = p$, we obtain for all $(q, p) \in \Gamma$

$$P_{qp} = Q_{qp}$$

that is

$$e = f$$

as claimed.

Note furthermore that bounded operators $A \in B(H)$ can also be represented by operator-valued kernels according to the prescription (see also Schroeck, 1982)

$$A \mapsto \hat{A}_e(q, p; q', p') = cP_{qp}AP_{q'p'} \tag{71}$$

and that

$$\int_\Gamma \hat{A}_e(q, p; q', p')(\hat{M}_e\psi)(q', p')\, dq'\, dp' = c^{3/2}\int_\Gamma P_{qp}AP_{q'p'}\psi\, dq'\, dp'$$

$$= c^{1/2}P_{qp}A\psi = (\hat{M}_eA\psi)(q, p)$$

so that

$$(\psi, A\psi) = (\hat{M}_e\psi, \hat{M}_eA\psi)_{L^2(\Gamma, H)}$$

$$= \int_\Gamma ((\hat{M}_e\psi)(q, p), \hat{A}_e(q, p; q', p')(\hat{M}_e\psi)(q', p'))\, dq\, dp\, dq'\, dp'$$

In particular, any quantum-mechanical *proposition* P (being, by definition, an orthogonal projector in the Hilbert space H) can be represented by an operator-valued integral kernel

$$cP_{qp}PP_{q'p'} = \hat{P}_e(q, p; q', p')$$

or, equivalently, by an integral operator on $L^2(\Gamma, H)$ defined by

$$(\mathbb{P}_e^P\Phi)(q, p) = \int_\Gamma \hat{P}_e(q, p; q', p')\Phi(q', p')\, dq'\, dp' \tag{72}$$

where $\Phi \in L^2(\Gamma, H)$.

$\mathbb{P}_e^P$ is obviously a projection operator, since the kernel $\hat{P}_e(q, p; q', p')$ is easily seen to be reproducing, i.e.,

$$\hat{P}_e(q, p; q', p') = \int_\Gamma \hat{P}_e(q, p; q'', p'')\hat{P}_e(q'', p''; q', p')\, dq''\, dp''$$

and

$$\hat{P}_e^*(q,p;q',p') = \hat{P}_e(q',p';q,p)$$

where * stands for the Hermitian conjugation.

Moreover, it can be easily shown that $\mathbb{P}_e^P$ projects onto the subspace $\hat{M}_e(P(H))$:

$$\mathbb{P}_e^P(L^2(\Gamma,H)) = \hat{M}_e(P(H)) \tag{73}$$

Of course, owing to the correspondence

$$P \mapsto P(H) \mapsto \hat{M}_e(P(H)) = \mathbb{P}_e^P(L^2(\Gamma,H)) \mapsto \mathbb{P}_e^P$$

we may expect that the mapping $P \mapsto \mathbb{P}_e^P$ preserves the ortholattice structure of the "*logic of propositions*" associated with a physical system described within the framework of the Hilbert space H. (We recall that the abovementioned "logic of propositions" consist of, by definition, all the orthoprojectors or, equivalently, the closed subspaces of the Hilbert space H.) This fact can be directly proven as follows (Q and P are here two arbitrary orthoprojectors in H):

$$\begin{aligned}
&(\mathbb{P}_e^Q\mathbb{P}_e^P\Phi)(q,p)\\
&\quad= \int_\Gamma \hat{Q}_e(q,p;q',p')\hat{P}_e(q',p';q'',p'')\Phi(q'',p'')\,dq'\,dp'\,dq''\,dp''\\
&\quad= c^2\int_\Gamma P_{qp}QP_{q'p'}PP_{q''p''}\Phi(q'',p'')\,dq'\,dp'\,dq''\,dp''\\
&\quad= c\int_\Gamma P_{qp}QPP_{q''p''}\Phi(q'',p'')\,dq''\,dp''
\end{aligned}\tag{74}$$

where the last equality is derived with the help of (63), and hence we easily get that $P \leqslant Q$ implies $\mathbb{P}_e^P \leqslant \mathbb{P}_e^Q$ and $P \perp Q$ implies $\mathbb{P}_e^P \perp \mathbb{P}_e^Q$.

To prove the converse implications, insert

$$\Phi = \hat{M}_e\psi, \qquad \psi \in H$$

into (74). Then

$$\begin{aligned}
(\mathbb{P}_e^Q\mathbb{P}_e^P\Phi)(q,p) &= c^{3/2}\int_\Gamma P_{qp}QPP_{q''p''}\psi\,dq''\,dp''\\
&= c^{1/2}P_{qp}QP\psi = [\hat{M}_e(QP\psi)](q,p)
\end{aligned}$$

So, if for example $\mathbb{P}_e^Q \mathbb{P}_e^P = \mathbb{P}_e^P$, then one must have

$$\hat{M}_e(QP\psi) = \hat{M}_e(P\psi)$$

and hence

$$QP\psi = P\psi$$

The equality above, valid for all ψ in H, shows that $P \leqslant Q$, and we similarly prove that $\mathbb{P}_e^P \perp \mathbb{P}_e^Q$ implies $P \perp Q$.

Note finally that if $P \perp Q$, then we obviously have

$$\mathbb{P}_e^{P+Q} = \mathbb{P}_e^P + \mathbb{P}_e^Q$$

If, in particular, $P + Q = I$, then

$$\mathbb{P}_e^P + \mathbb{P}_e^Q = \mathbb{P}_e^I = \mathbb{P}_e$$

In other words,

$$\mathbb{P}_e^{P'} = \mathbb{P}_e - \mathbb{P}_e^P = \left(\mathbb{P}_e^P\right)'$$

We thus see that the map $P \mapsto \mathbb{P}_e^P$ preserves orthocomplementation too.

We shall now pass on to the case of an *atomic* identity resolution $e = \{P_{qp}\}$ satisfying (63). The passage to a phase-space representation of quantum mechanics by using an atomic resolution of the identity was studied in great detail by Ali and Prugovečki (1977b) and it was based on the correspondence

$$\psi \mapsto M_e\psi \in L^2(\Gamma), \qquad \psi \in H \tag{75}$$

defined by

$$(M_e\psi)(q, p) = c^{1/2}(e_{qp}, \psi) \tag{76}$$

where e_{qp} denotes a fixed norm one vector in the range $P_{qp}(H)$ of the one-dimensional projector P_{qp} belonging to the identity resolution e.

Obviously,

$$\begin{aligned}(\hat{M}_e\psi)(q, p) &= c^{1/2}P_{qp}\psi = c^{1/2}(e_{qp}, \psi)e_{qp} \\ &= (M_e\psi)(q, p)e_{qp}\end{aligned}$$

so that our former $\hat{M}_e$ [see (64) and (65)] can now be obtained from M_e. Of course, M_e can be derived from $\hat{M}_e$ too with the help of the map $\tilde{M}_e$: $H \to L^2(\Gamma, H)$ defined by

$$(M_e\psi)(q,p) = \begin{cases} \|(\hat{M}_e\psi)(q,p)\|^{-1}(\hat{M}_e\psi)(q,p) & \text{if } (\hat{M}_e\psi)(q,p) \neq 0 \\ 0 & \text{if } (\hat{M}_e\psi)(q,p) = 0 \end{cases}$$

We have

$$(M_e\psi)(q,p) = c^{1/2}(e_{qp},\psi) = \|(\hat{M}_e\psi)(q,p)\|\big(e_{qp},(\tilde{M}_e\psi)(q,p)\big)$$

So, we conclude that the *two passages to a phase-space representation of quantum mechanics described in* (65) *and* (76), *respectively, are in fact equivalent, if we restrict ourselves to the case of an atomic resolution of the identity.*

It is not difficult to show (Ali and Prugovečki, 1977b) that the map $\psi \mapsto M_e\psi$ is an *isometry* from H into $L^2(\Gamma)$, and that the orthogonal projector $\mathbb{P}_e$ of $L^2(\Gamma)$ onto $M_e(H)$ is of the form

$$(\mathbb{P}_e\Phi)(q,p) = \int_\Gamma K_e(q,p;q',p')\Phi(q',p')\,dq'\,dp', \qquad \Phi \in L^2(\Gamma)$$

with the kernel

$$K_e(q,p;q',p') = c(e_{qp}, e_{q'p'})$$

which obviously has the following properties:

$$K_e(q,p;q',p') = \int_\Gamma K_e(q,p;q'',p'')K_e(q'',p'';q',p')\,dq''\,dp''$$

$$\bar{K}_e(q,p;q',p') = K_e(q',p';q,p)$$

where the bar stands for the complex conjugation.

The bounded operators A in $B(H)$ become represented by integral kernels

$$A_e(q,p;q',p') = c(e_{qp}, Ae_{q'p'})$$

so that

$$(\psi, A\psi) = (M_e\psi, M_eA\psi)_{L^2(\Gamma)}$$

$$= \int_\Gamma \overline{(M_e\psi)}(q,p)A_e(q,p;q',p')(M_e\psi)(q',p')\,dq\,dp\,dq'\,dp'$$

In particular, the orthogonal projectors P in H are represented by kernels

$$P_e(q, p; q', p') = c(e_{qp}, Pe_{q'p'})$$

or, equivalently, by integral operators on $L^2(\Gamma)$ defined by

$$(\mathbb{P}_e^P\Phi)(q, p) = \int_\Gamma P_e(q, p; q', p')\Phi(q', p')\, dq'\,dp', \qquad \Phi \in L^2(\Gamma)$$

and we can show, as before, that $\mathbb{P}_e^P$ are projection operators, and that the correspondence

$$P \mapsto \mathbb{P}_e^P$$

is an orthoinjection of the ortholattice of the orthogonal projectors in H into the corresponding ortholattice in the Hilbert space $L^2(\Gamma)$.

Clearly, we can also prove that the correspondence

$$e \mapsto \mathbb{P}_e$$

between the atomic resolutions of the identity on H and the corresponding orthogonal projectors $\mathbb{P}_e$ in $L^2(\Gamma)$ is one-to-one.

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